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HEAT CONDUCTION IN A SEMI-INFINITE SOLID WITH VARIABLE THERMOPHYSICAL PROPERTIES

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NOMENCLATURE

C, thermal capacity at t;
C_0, thermal capacity at t_0;
erf, error function;
erfc, complimentary error function;
k, thermal conductivity at t;
k_0, thermal conductivity at t_0;
t, temperature;
x, space coordinate;
y, z, similarity variables.

Greek symbols
α, thermal diffusivity;
ε, physical quantities defined in equations (3) and (11) used as perturbation parameters for Case I and Case II respectively;
τ, time;
ρ, density;
ϕ, dimensionless temperature.

Subscripts
0, 1, 2, zero-, first-, and second-order perturbation relations for Case I;
00, zero-order relation for Case II;
01, 10, first-order relations for two parameters ε_1 and ε_2 expansion in Case II.

1. INTRODUCTION

THE CLASSICAL problem of transient heat conduction in a semi-infinite solid having variable thermal conductivity, with a stepwise temperature change at the surface is discussed in [1–3]. Yang [1] solved this numerically by the method of successive approximations; whereas Goodman [2] and Vujanovic [3] employed integral and optimal linearization methods respectively. This report presents the analysis of above problem and also the simultaneous variation of thermal conductivity and thermal capacity, using perturbation techniques, which after successive integration, though involved for higher orders, give closed form analytic solutions in both cases.

2. STATEMENT AND SOLUTION OF THE PROBLEM

Case I: *Transient heat conduction in a semi-infinite solid having variable thermal conductivity*

An isotropic semi-infinite slab initially at temperature t_i, is suddenly raised to temperature t_0 at its surface for time $\tau > 0$. Introducing a new independent variable $y = x/2(\alpha\tau)^{1/2}$ and $\phi = (t-t_i)/(t_0-t_i)$, the governing heat conduction equation and boundary conditions are as in [1],

$$K^+\frac{d^2\phi}{dy^2}+\frac{dK^+}{d\phi}\left[\frac{d\phi}{dy}\right]^2+2y\frac{d\phi}{dy}=0 \qquad (1)$$

$$\phi(0)=1;\quad \phi(\infty)=0. \qquad (2a, 2b)$$

The variation of thermal conductivity with temperature is

$$K^+ = k/k_0 = 1+\varepsilon\phi \qquad (3)$$

where ε is small and numerically less than one for most metals and alloys. This smallness suggests an asymptotic expansion for ϕ in terms of ε as

$$\phi = \phi_0+\varepsilon\phi_1+\varepsilon^2\phi_2+O(\varepsilon^3). \qquad (4)$$

Substituting equations (3) and (4) in equations (1) and (2) and equating like powers of ε, results in the following set of equations for ϕ_0, ϕ_1 and ϕ_2 with appropriate boundary conditions, where primes denote differentiation with respect to y.

$$\phi_0''+2y\phi_0'=0;\quad \phi_0(0)=1,\ \phi_0(\infty)=0 \qquad (5)$$

$$\phi_1''+2y\phi_1'+\phi_0\phi_0''+\phi_0'^2=0 \qquad (6)$$

$$\phi_1(0)=0;\quad \phi_1(\infty)=0$$

$$\phi_2''+2y\phi_2'+\phi_0\phi_1''+\phi_1\phi_0''+2\phi_0'\phi_1'=0$$

$$\phi_2(0)=0;\quad \phi_2(\infty)=0. \qquad (7)$$

The solution for the zero-order problem which corresponds to the constant conductivity medium, is given by

$$\phi_0(y)=\mathrm{erfc}\,(y)=1-\frac{2}{(\pi)^{1/2}}\cdot\int_0^y e^{-u^2}du. \qquad (8)$$

The solutions $\phi_1(y)$ and $\phi_2(y)$ which can be obtained either by direct integration or by the method of variation of parameters, are presented in terms of ϕ_0 as follows:

$$\phi_1(y)=\left[\frac{1}{2}+\frac{1}{\pi}\right]\phi_0-\frac{\phi_0^2}{2}-\frac{1}{\pi}e^{-2y^2}+\frac{1}{(\pi)^{1/2}}ye^{-y^2}\phi_0 \qquad (9)$$

$$\begin{aligned}\phi_2(y)=&\left[-\frac{1}{\pi}+\frac{2}{\pi^2}+\frac{3(3)^{1/2}}{4\pi}\right]\phi_0-\left(\frac{1}{2}+\frac{1}{\pi}\right)\phi_0^2+\frac{\phi_0^3}{2}\\&+\frac{1}{2\pi(\pi)^{1/2}}ye^{-3y^2}+\frac{1}{\pi}\left[(3-y^2)\phi_0-1-\frac{2}{\pi}\right]e^{-2y^2}\\&+\frac{1}{4(\pi)^{1/2}}\left[\left(4+\frac{8}{\pi}\right)y\phi_0-(9y-2y^3)\phi_0^2\right]e^{-y^2}\\&+\frac{3(3)^{1/2}}{4\pi}\,\mathrm{erfc}\,[y(3)^{1/2}].\end{aligned} \qquad (10)$$

The complete three term perturbation solution is formed by substituting equations (8)–(10) in equation (4).

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Case II: Transient heat conduction in a semi-infinite solid having both variable thermal capacity and thermal conductivity

Here both thermal capacity and thermal conductivity of the medium vary with temperature in the following form:

$$c/c_0 = 1+\varepsilon_1\phi; \quad k/k_0 = 1+\varepsilon_2\phi. \tag{11}$$

Using a new independent variable $z = (x/2)(c_0\rho/k_0\tau)^{1/2}$, the governing equation and its boundary conditions become:

$$2z\frac{c}{c_0}\frac{d\phi}{dz}+\frac{d}{dz}\left[\frac{k}{k_0}\frac{d\phi}{dz}\right]=0$$
$$\phi(0)=1, \quad \phi(\infty)=0. \tag{12}$$

Considering ε_1 and ε_2 as small and numerically less than one, the asymptotic expansion for ϕ in terms of two parameters ε_1 and ε_2 is

$$\phi(z) = \phi_{00}+\varepsilon_1\phi_{01}+\varepsilon_2\phi_{10}+O(\varepsilon_1\varepsilon_2). \tag{13}$$

Following the same procedure as in Case I, results in the following system of equations with corresponding boundary conditions:

$$\phi''_{00}+2z\phi'_{00} =$$
$$\phi_{00}(0)=1; \quad \phi_{00}(\infty)=0 \tag{14}$$

$$\phi''_{01}+2z[\phi'_{01}+\phi_{00}\phi'_{00}]=0$$
$$\phi_{01}(0)=0=\phi_{01}(\infty) \tag{15}$$

$$\phi''_{10}+2z\phi'_{10}+\phi_{00}\phi''_{00}+\phi'^2_{00}=0$$
$$\phi_{10}(0)=0=\phi_{10}(\infty). \tag{16}$$

The solutions for equations (14) and (16) are the same as given in equations (8) and (9) respectively, if the variable y is replaced by z. The solution for equation (15) perturbed in ε_1, given in terms of the zero-order solution is:

$$\phi_{01} = \tfrac{1}{2}z\phi_{00}\phi'_{00}-\frac{\phi_{00}}{\pi}+\frac{\phi'^2_{00}}{4}. \tag{17}$$

3. RESULTS AND DISCUSSION

In Fig. 1, the perturbation solution upto second-order for conductivity variation is compared with that of Yang [1] for values of $\varepsilon = 0.5$ and -0.5. The present solution is in explicit closed form whereas in [1] successive approximation is employed and in [2] the accuracy of the integral solution relies on refinement of the temperature profile. In the optimal linearization method [3], the solution is also dependent on the "supposed function" for the temperature profile. Figure 1 also shows the effect of simultaneous variation of both capacity and conductivity on temperature distribution for $\varepsilon_1 = \varepsilon_2 = -0.5$ and 0.5 respectively. Table 1 compares the values of heat-transfer rate at the surface obtained from the different methods. The heat flow rate into the solid obtained from the perturbation solution shown in the last column of Table 1 has an accuracy of $O(\varepsilon^3)$. It is to be noted that Yang's solution is most accurate; however, the asymptotic solutions may be taken as very close to the exact solution for $\varepsilon \ll 1$ due to the rapid decrease in magnitude of the first few coefficients in the solution.

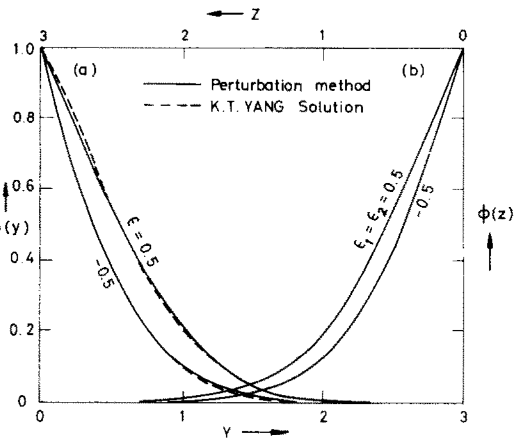

FIG. 1. Temperature distributions in a semi-infinite solid with; (a) variable thermal conductivity; (b) variable thermal conductivity and thermal capacity.

4. CONCLUSION

Perturbation solutions are obtained for the temperature distribution of a medium whose thermophysical properties vary with temperature. Zero-order solutions correspond to a medium of constant thermo-physical properties. Though the expansion is valid for $\varepsilon < 1$, the results in this report can be adopted as reasonable approximation for engineering applications.

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Table 1. Comparison of heat flow rate into the solid having variable thermal conductivity, for different methods. Values of $-\phi'(0)$

ε	K. T. Yang	Integral cubic	Method quartic	Optimal linearization method	Present solution upto I order	II order
0.5	0.863	1.00	0.865	0.936	0.7434	0.9230
0.0	1.128	1.23	1.100	—	1.1280	1.1280
−0.5	1.859	1.74	1.790	1.524	1.5125	1.6914